- Why are Fourier Series useful? 1) to represent discontinuous periodic functions with continuous periodic functions. 2) for boundary value problems (BVPs)
  - Examples of discontinuous periodic functions: square wave, triangle wave, sawtooth wave
- **Orthogonality**: Let *f* and *g* be periodic functions, both with period *T*. *f* and *g* are orthogonal on an interval $I=[-L,L]$ if $\int_{-L}^{L} f(x)g(x)dx = 0$.
  - Combinations of sines with cosines (i.e. $f(x)=A_1\cos(c_1x)$ and $g(x)=A_2\sin(c_2x)$) are always orthogonal. Proof: $f(x)=A_1\cos(c_1x)$ is an even function, $g(x)=A_2\sin(c_2x)$ is an odd function. The product of an even function and an odd function is an odd function. The definite integral of an odd function on $[-L,L]$ is always 0.
- **Fourier Series**
  - Let $f(x)$ be a periodic function with period $2L$, let $F(x)$ denote its Fourier series. $F(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right)$, whereas
    - $a_0=\frac{1}{L}\int_{-L}^{L} f(x)dx$
    - $a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$
    - $b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$
  - Alternatively: $f(x)=\sum_{n=-\infty}^{\infty}c_ne^{\frac{ni\pi x}{L}}$, where $c_n=\frac{1}{2L}\int_{-L}^{L} f(x)e^{\frac{-ni\pi x}{L}}dx$.
  - Proof of this formula using orthogonality and integrating the product of combinations of sines and cosines.
  - These calculations can be quite tedious. So here are a few shortcuts:
    - If $f(x)$ is odd, then the series will only contain the sine terms (i.e. $a_n=0$).
    - If $f(x)$ is even, then the series will not contain any sine terms (i.e. $b_n=0$).
    - These are called the **Fourier sine series** and **Fourier cosine series** respectively.
  - Theorem: If $f(x)$ is continuous at x_o, then $f(x_o)=F(x_o)$. If $f(x)$ is discontinuous at x_o, then $F(x_o)=\frac{1}{2}\left(\lim_{x\to x_o^+} f(x)+\lim_{x\to x_o^-} f(x)\right)$.